

Consistent boundary conditions for supergravity

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Abstract

We derive the complete orbit of boundary conditions for supergravity models which is closed under the action of all local symmetries of these models, and which eliminates spurious field equations on the boundary. We show that the Gibbons-Hawking boundary conditions break local supersymmetry if one imposes local boundary conditions on all fields. Nonlocal boundary conditions are not ruled out. We extend our analysis to BRST symmetry and to the Hamiltonian formulation of these models.

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1 Introduction

Supergravity is supersymmetric general relativity. When it was first constructed, as a field theory in $3 + 1$ dimensions with $N = 1$ gravitino, partial integrations in the proof of local supersymmetry were performed without taking boundary terms into consideration [1]. However, it was clear that in the presence of boundaries, local supersymmetry (and other local symmetries such as local coordinate invariance and local Lorentz symmetry¹) can only remain unbroken if one imposes certain boundary conditions on the fields and on the parameters.

A natural arena to describe any theory with supersymmetry (called henceforth susy) is superspace. One usually begins by defining the integration over the anticommuting coordinates θ^α as ordinary Grassmann integration, $\int d\theta^\alpha$ with $\int d\theta^\alpha \theta^\alpha = 1$ but $\int d\theta^\alpha 1 = 0$. A more convenient way to identify the x -space component fields contained in superfields is to use susy covariant derivatives $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu$, and to replace $\int dx d\theta^\alpha$ by $\int dx D_\alpha$. As long as one may drop total x -derivatives, this makes no difference, but it is clear that in the presence of boundaries the results of these two approaches differ by boundary terms. These boundary terms can be described by a boundary superspace [2].

For ordinary gravity ($N = 0$ supergravity), York [3], and Gibbons and Hawking [4], established long ago that one can cancel most of the boundary terms which one obtains if one varies the metric in the Einstein-Hilbert action by adding a boundary term which contains the extrinsic curvature of the boundary. A completely arbitrary variation of the metric in the sum of the Einstein-Hilbert bulk action and the boundary action yields then the following result

$$\delta\mathcal{S}_{\text{EH}} + \delta\mathcal{S}_{\text{bound}} = \int_{\mathcal{M}} G^{\mu\nu} \delta g_{\mu\nu} + \int_{\partial\mathcal{M}} (K^{\mathbf{ij}} - g^{\mathbf{ij}} K_{\mathbf{k}}^{\mathbf{k}}) \delta g_{\mathbf{ij}}. \quad (1)$$

Here $G^{\mu\nu}$ denotes the Einstein tensor, \mathcal{M} is the manifold with the usual measure $\sqrt{|\det g_{\mu\nu}|}$, $\partial\mathcal{M}$ its boundary with measure $\sqrt{|\det g_{\mathbf{ij}}|}$, $K^{\mathbf{ij}}$ is the extrinsic curvature (see appendix A), and indices μ, ν refer to coordinates x^μ in the bulk, while indices \mathbf{i}, \mathbf{j} refer to coordinates $x^{\mathbf{i}}$ in the boundary. We consider a boundary of dimension $D - 1$ if spacetime has D dimensions.

¹Local Lorentz symmetry is an internal symmetry, but boundary conditions on spinors which are not Lorentz invariant may lead to boundary conditions on the Lorentz parameters.

Our results apply equally well to boundaries in time and to boundaries in space. We denote the coordinate which leads away from the surface by t , even though it may be a spacelike coordinate. Gibbons and Hawking proposed to impose the boundary condition that the variations of the metric in the surface vanish

$$\delta g_{ij}|_{\partial\mathcal{M}} = 0. \quad (2)$$

We shall demonstrate that this violates local susy if one only admits local boundary conditions (in particular no boundary conditions on curvature components but only on the fields themselves).

We consider the case with zero cosmological constant only. For a non-zero cosmological constant, the action on shell is infinite, but one can add a boundary term to make it finite [5]. The cosmological constant also brings an additional dimensional parameter in the bulk action which can be used to construct boundary actions. An example of such actions is the “boundary cosmological constant” which appears in supergravity theories with a bulk cosmological constant (see [6, 7]).

The aim of the present article is to determine boundary conditions (called BC henceforth) for pure supergravity theories which maintain all local symmetries. Of course, not only the fields but also the gauge parameters of these symmetry transformations must then be restricted on the boundary because gauge parameters become ghost fields in the BRST formalism. It is important, that they are not over-restricted. (For example, imposing both Dirichlet and Neumann BC on the same component of a field or a parameter simultaneously clearly over-restricts this field or parameter). In the BRST approach one requires the BRST invariance of the BC [8, 9]. Several authors have already tackled many aspects of this problem, in particular D’Eath, Luckock, Moss and Esposito [10, 6, 11, 12]. Luckock and Moss [6] have found an extra fermionic term in the action on the boundary such that the whole action (bulk action plus boundary action) is locally susy under variation with a restricted susy parameter. They imposed the BC $\delta g_{ij} = 0$, but we shall pursue the question whether this condition is part of a full set of BC which close under local susy (see below). Even studies have appeared with nonlinear sigma models on manifolds with boundaries [6, 13]. We shall instead consider pure supergravities, without extra fields. We consider both models with auxiliary fields and models without them. Our aim is to derive a complete and consistent set of BC. By this we mean the following. The total set of BC should satisfy the following two requirements

- (i) it should not produce extra field equations on the boundary. For example, we shall derive that one of the BC on the fields themselves is $K^{\mathbf{ij}} = 0$, which evidently is the alternative “Neumann BC” for gravity, instead of the “Dirichlet BC” $\delta g_{\mathbf{ij}} = 0$ for the variations in (1). ($K^{\mathbf{ij}} = -\frac{1}{2}\partial_{\mathbf{n}}g^{\mathbf{ij}}$ in Gaussian coordinates, where $\partial_{\mathbf{n}}$ is the normal derivative). We impose these BC on off-shell fields, even though they are derived from an analysis of the field equations. Our aim is to use the total set of BC to define the space of fields we consider both on- and off-shell. So we do not want to begin with a set of BC for off-shell fields, and then later impose separate BC for on-shell fields.
- (ii) any rigid or local symmetry of the theory should transform any BC into a linear combination of BC. For example, we shall require that also the local susy variation of $K^{\mathbf{ij}}$ vanishes, and this will yield new BC. At the quantum level we shall replace the set of all local symmetries by BRST symmetry, but the same requirement will be imposed. This leads to BC on the ghosts, as is well-known in string theory. It may seem contradictory to the non-expert that one gets different BC from field equations or local symmetries, because any local symmetry variation can always be written as field equations times the local symmetry variation of the fields. However, in order that variations cancel against each other one needs further partial integrations which lead to further BC. This is well-known among supergravity practitioners.

Before moving on, we should be clear about whether we impose BC on background fields or on fluctuations about the background fields. We only consider bosonic background fields. We study first in section 2 a trivial (flat space) background in which case there are only fluctuating fields, and all BC refer to these fields. When one considers background fields (for example, AdS space, or a black hole), we require that these background fields are susy, meaning that the susy transformation rules of the fermions vanish if one substitutes the background fields into the right-hand sides of these transformation rules. This leaves in general only rigid susy transformations with susy parameters whose spacetime dependence is fixed. Given such susy backgrounds, the transformation rules reduce to rigid transformations for the fluctuation fields. However, even under this restricted class of rigid susy variations the classical action is not in general invariant because boundary terms may remain. Then BC for the background fields like $K^{\mathbf{ij}} = 0$ can in general not be met, but only special boundaries (for example boundaries spanned by

geodesics) can achieve this. Alternatively, one can try to add boundary terms to the action such that the BC for the background fields become satisfied. In section 3 we give a simple example of such boundary terms. The set of all fluctuating fields satisfying the complete set of BC forms a linear vector space, and the consistency of the BC means that symmetry transformations never lead one out of this space.

Our strategy is as follows. We view the total set of BC as an orbit, and we shall move freely forward and backward along this orbit, postponing the solution of difficult constraints until we have obtained more information from other constraints which are easier to solve. For example, one may use information from local susy, in particular the closure of the gauge algebra, to solve explicitly the equation $\delta K^{\mathfrak{U}} = 0$. Since it still is true that supergravity is less well-known than general relativity, we shall be very explicit and illustrate our results with a simple supergravity theory, $N = 1$ supergravity in $2 + 1$ dimensions. (Real Majorana spinors for $N = 1$ theories require Minkowski spacetime instead of Euclidean space). Another strategy we shall pursue is that we view those BC which are needed to preserve local symmetries of the action as kinematical, in the sense that they should not depend on the dynamics of a particular model. So, for example, we may study pure gravity, to learn about the BC on the metric, its variations, and the diffeomorphism parameters. Then we may use these results in supergravity, for example requiring that the composite parameter $\xi^\mu = \bar{\epsilon}_2 \gamma^\mu \epsilon_1$ in local gauge algebra satisfies the same BC as ξ^μ in general relativity. A very simple way to derive a subset of all BC is to consider a special case: free field theories with rigid supersymmetries (section 2). In section 3 we consider a model with a background: the susy kink. As an amusing warming up exercise for the full nonlinear supergravities we consider in section 4 a quantum mechanical model for supergravity where all nonlinearities have a simple structure. In section 5 we consider BC in a Hamiltonian version of this model, and compare our results to those obtained from the BRST formalism. In section 6 we apply our insights to the simple supergravity model mentioned above, and in section 7 we draw conclusions.

In this paper we restrict our attention to local BC. Namely, for any (multicomponent) field ϕ we define two complementary local projectors on the boundary, P_D and P_N , $P_D + P_N = 1$ such that $P_D \phi$ satisfies the Dirichlet BC $P_D \phi|_{\partial \mathcal{M}} = 0$, and $P_N \phi$ satisfies the modified Neumann (Robin) BC $(\partial_{\mathbf{n}} + \mathcal{S})P_N \phi|_{\partial \mathcal{M}} = 0$, where \mathcal{S} is a matrix-valued function on the boundary. Our analysis even allows for \mathcal{S} containing derivatives along the boundary up

to a finite order (as in the theory of open strings) although such derivative terms do not appear in the particular models considered below. Such BC with a non-derivative \mathcal{S} are called mixed BC. More about general properties of mixed BC one can learn from [14, 15, 16]. This restriction to local BC looks rather natural. Indeed, the conditions we impose on the fields at a given point of the boundary must not depend on the value of the same field at distant points. Several authors have used nevertheless nonlocal BC for supergravity [17, 18, 12], but no closed locally supersymmetric set (orbit) of such BC was found. In the present paper we show that the Gibbons-Hawking BC (2) cannot be extended to a consistent locally susy orbit of *local* BC. This result may indicate that one has to reconsider nonlocal BC in supergravity.

The supergravity community has not studied BC in detail in the past, but the advent of string theory where BC play a crucial role may also lead to further work on BC in supergravity from the same perspective as in string theory². Our article follows the same approach as an earlier article by Lindström, Rocek and one of us [2] on BC in superstring theory, and also in a paper by the other author [19] on the susy vortex. Several of our results confirm results on BC in supergravity theories by others, and we shall try to give references whenever this is possible. However, we believe that the complete orbit of BC is new.

2 Linearized supergravity with rigid susy

A simple and direct way to obtain (some) BC on fields and local parameters is to consider linearized field theories with rigid parameters. Thus we consider in this section linearized supergravity in $3 + 1$ dimensions. The number of space-time dimensions is not crucial. After obvious modifications the results of this section will be valid, for example, also in $2 + 1$ dimensions. There are no background fields, so all BC are on the fluctuating fields. This analysis reveals the existence of two sets of BC, one with $K^{ij} = 0$, the other with $\delta g_{ij} = 0$. In section 6 we shall consider the consequences of extending the analysis to full nonlinear local susy, and we shall find that only the set with $K^{ij} = 0$ is consistent in the sense described in the introduction.

It would seem natural to start with the linearized spin 2 and spin 3/2 fields, but their analysis is rather complicated, and for that reason we start

²In several studies of the AdS/CFT correspondence, BC on fields play a crucial role [5], but the invariance of these BC under local susy has not been studied.

at the other end, with the spin 0 - spin 1/2 system and gather rather efficiently information which will be of use for the spin 2 - spin 3/2 system. Some of the statements regarding rigid susy of BC which we derive below are already known and collected in [18].

The action and susy transformations for a system consisting of a scalar S , a pseudoscalar P and a Majorana spinor λ in four dimensions read

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}(\partial_\mu P)^2 - \frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda, \\ \delta_\epsilon S &= \bar{\epsilon}\lambda, \quad \delta_\epsilon P = i\bar{\epsilon}\gamma_5\lambda, \quad \delta_\epsilon\lambda = (\gamma^\mu\partial_\mu S + i\gamma^\mu\partial_\mu P\gamma_5)\epsilon.\end{aligned}\tag{3}$$

We are in Minkowski space. In our conventions $\gamma_5^2 = 1$, $\gamma_5^\dagger = \gamma_5$ and the γ^j (with j a space-like index) are hermitian, while γ^0 is antihermitian. The symbol $\bar{\lambda}$ denotes the Dirac conjugate $\lambda^\dagger i\gamma^0$ (which is equal to the Majorana conjugate $\lambda^T C$ for a Majorana spinor, with C the charge conjugation matrix). The action is real, and the susy transformation rules preserve the reality properties of the fields. The matrix $\gamma^n = \gamma_\mu n^\mu$ with n^μ the normal to the boundary has no definite reality properties; in special cases it can be hermitian or antihermitian, but all formulas derived in the text hold for all cases, essentially because $\lambda^\dagger(\gamma^n)^\dagger i\gamma^0 = -\bar{\lambda}\gamma^n$. The Euler-Lagrange variation of the fields in the action (3) leads to the boundary terms $-\delta S\partial^n S - \delta P\partial^n P - \frac{1}{2}\bar{\lambda}\gamma^n\delta\lambda$, and following standard arguments of string theory one concludes that there are four possibilities in the spin 0 sector: Dirichlet conditions ($S|_{\partial\mathcal{M}} = 0$, $P|_{\partial\mathcal{M}} = 0$) or Neumann conditions ($\partial^n S|_{\partial\mathcal{M}} = 0$, $\partial^n P|_{\partial\mathcal{M}} = 0$) for S and P . One could add a boundary term $\mathcal{L}_{\partial\mathcal{M}} = S\partial^n S$ (and a similar term for P). Then one would be left with the boundary variation $S\partial^n\delta S$ instead of $\delta S\partial^n S$ and the same BC would be obtained³. On the boundary one can at most restrict one half of the spinor variables. We need BC for λ without derivatives to cancel $-\frac{1}{2}\bar{\lambda}\gamma^n\delta\lambda$. This leads to one of the following BC on λ : either $P_+\lambda|_{\partial\mathcal{M}} = 0$, or $P_-\lambda|_{\partial\mathcal{M}} = 0$, where P_+ and P_- are projection operators [21]

$$P_\pm = \frac{1}{2}(1 \pm \gamma^n).\tag{4}$$

There is now no boundary term which can cancel (part of) the boundary variation $-\frac{1}{2}\bar{\lambda}\gamma^n\delta\lambda$ since $\bar{\lambda}\gamma^n\lambda$ vanishes.

³In ref. [20] a boundary term $-\mu/2\phi\partial_n\phi - \kappa/2\phi^2$ is considered, and renormalization effects produce counterterms proportional to μ . A discontinuous field redefinition in their eq. (2.5) $\phi(x, y) \rightarrow \phi(x, y) + \alpha\theta(y_0 - y)\phi(x, y_0)$ with y_0 on the boundary, and a coupling constant redefinition involving $\delta(0)$ absorbs μ .

The bulk field equations combined with these BC lead to a tower of further BC involving even numbers of derivatives for the bosons and powers of $P_+ \partial_{\mathbf{n}}$ for the fermions. For example, $S|_{\partial\mathcal{M}} = 0$ and $\square S = 0$ leads to $\partial_{\mathbf{n}}^2 S|_{\partial\mathcal{M}} = 0$, $\partial_{\mathbf{n}}^4 S|_{\partial\mathcal{M}} = 0$ etc, while $P_- \lambda|_{\partial\mathcal{M}} = 0$ is accompanied⁴ by $P_+ \partial_{\mathbf{n}} \lambda|_{\partial\mathcal{M}} = 0$ etc. However, when we discuss the invariance of BC under symmetries of the action, we shall not require that the fields satisfy their field equations.

In the presence of boundaries, one half of the susy is always violated. Indeed, consider BC for the scalar. If one takes, for example, $\delta S|_{\partial\mathcal{M}} = 0$, consistency requires that $\delta_\epsilon S|_{\partial\mathcal{M}} = \bar{\epsilon} \lambda|_{\partial\mathcal{M}} = 0$. If $P_\pm \lambda|_{\partial\mathcal{M}} = 0$, one has to impose $\bar{\epsilon} P_\mp = 0$ because $\bar{\epsilon} \lambda = \bar{\epsilon} (P_+ + P_-) \lambda$. We suppose, that unbroken susy always corresponds to $\bar{\epsilon} P_+ = 0$, which is equivalent to

$$P_- \epsilon = 0. \quad (5)$$

The opposite choice ($P_+ \epsilon = 0$) leads to the equivalent results. It is easy to see that there are two sets of BC which are invariant under the susy transformations with the parameter restricted according to (5)

$$S|_{\partial\mathcal{M}} = 0, \quad \partial^{\mathbf{n}} P|_{\partial\mathcal{M}} = 0, \quad P_- \lambda|_{\partial\mathcal{M}} = 0 \quad (6)$$

or

$$P|_{\partial\mathcal{M}} = 0, \quad \partial^{\mathbf{n}} S|_{\partial\mathcal{M}} = 0, \quad P_+ \lambda|_{\partial\mathcal{M}} = 0. \quad (7)$$

Susy variations of these conditions produce again BC with extra $\partial_{\mathbf{n}}$ derivatives, but now these BC are conditions for off-shell fields in the action. For example, consistency of the BC in (6) leads to the further set $\partial_{\mathbf{n}}^{2m} S|_{\partial\mathcal{M}} = 0$, $\partial_{\mathbf{n}}^{2m-1} P|_{\partial\mathcal{M}} = 0$, $P_- \partial_{\mathbf{n}}^{2m} \lambda|_{\partial\mathcal{M}} = 0$ and $P_+ \partial_{\mathbf{n}}^{2m+1} \lambda|_{\partial\mathcal{M}} = 0$ for $m = 1, 2, 3, \dots$

The contravariant index \mathbf{n} in $\partial^{\mathbf{n}} S$ is defined by $\partial_\mu (\delta S \partial^\mu S) = \partial_{\mathbf{n}} (\delta S \partial^{\mathbf{n}} S) + \partial_{\mathbf{i}} (\delta S \partial^{\mathbf{i}} S)$, where $\partial_{\mathbf{n}} = n^\mu \partial_\mu$ and $\partial_{\mathbf{i}} = \frac{\partial}{\partial x^{\mathbf{i}}}$. It is lowered by the Minkowski metric in the coordinate system with coordinates $(x^{\mathbf{n}}, x^{\mathbf{i}})$ along the normal and in the boundary.

Next we turn to the free spin 1 - spin 1/2 system

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda, \\ \delta_\epsilon A_\mu &= \bar{\epsilon} \gamma_\mu \lambda, \quad \delta_\epsilon \lambda = -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} \epsilon, \end{aligned} \quad (8)$$

⁴Consider the field equation $\gamma^\mu \partial_\mu \lambda = 0$. One finds by acting with P_+ that $\gamma^{\mathbf{n}} (P_+ \partial_{\mathbf{n}} \lambda) + \gamma^{\mathbf{j}} \partial_{\mathbf{j}} (P_- \lambda) = 0$. Hence if $P_- \lambda|_{\partial\mathcal{M}} = 0$ (and thus also $\partial_{\mathbf{j}} (P_- \lambda)|_{\partial\mathcal{M}} = 0$) then $P_+ \partial_{\mathbf{n}} \lambda|_{\partial\mathcal{M}} = 0$ on shell. In Euclidean space one has to choose different projection operators $P_\pm^{(E)} = \frac{1}{2} (1 \pm i \gamma^{\mathbf{n}} \gamma_5)$.

where $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. In general, local BC for spin 1 fields can either be magnetic ($A_j|_{\partial\mathcal{M}} = 0$) or electric ($F^{jn}|_{\partial\mathcal{M}} = 0$) [18]⁵. We split $F^{nj}|_{\partial\mathcal{M}} = 0$ into the stronger set of BC $A_n|_{\partial\mathcal{M}} = 0$ and $\partial_n A_j|_{\partial\mathcal{M}} = 0$. For both BC the boundary term $-\delta A_j F^{jn}$ produced by the Euler-Lagrange variation vanishes. There are three boundary terms possible for A_μ , namely $A_j F^{jn}$, $A_j \partial^n A^j$ and $A_n \partial^n A^n$. The first one leads to the boundary variation $A_j \delta F^{jn} + \delta A_j F^{jn}$, and again the BC are the same as without this boundary term. The boundary terms $A_j \partial^n A^j$ and $A_n \partial^n A^n$ are invariant under both sets of BC; in fact, they vanish since we shall soon show that $A_j|_{\partial\mathcal{M}} = 0$ implies that also $\partial_n A_n|_{\partial\mathcal{M}} = 0$. So there is no boundary term for spin 1 either. The other boundary term $-\frac{1}{2}\bar{\lambda}\gamma^n\delta\lambda$ we discussed before. However, only one of these BC is compatible with susy for each of the BC for the spin 1/2 field. As a result, the following two sets are susy invariant

$$S|_{\partial\mathcal{M}} = 0, \quad \partial^n P|_{\partial\mathcal{M}} = 0, \quad P_- \lambda|_{\partial\mathcal{M}} = 0, \quad A_n|_{\partial\mathcal{M}} = 0, \quad \partial_n A_j|_{\partial\mathcal{M}} = 0 \quad (9)$$

or

$$P|_{\partial\mathcal{M}} = 0, \quad \partial^n S|_{\partial\mathcal{M}} = 0, \quad P_+ \lambda|_{\partial\mathcal{M}} = 0, \quad A_j|_{\partial\mathcal{M}} = 0. \quad (10)$$

Susy variations of these BC lead again to BC with additional ∂_n derivatives as discussed above. For example, the susy variation of $P_+ \lambda|_{\partial\mathcal{M}} = 0$ in (10) leads to $F_{ij}|_{\partial\mathcal{M}} = 0$ which confirms $A_j|_{\partial\mathcal{M}} = 0$. One would expect also $\partial_n A_n|_{\partial\mathcal{M}} = 0$ in (10), and we shall indeed obtain this BC when we consider the spin 1 - spin 3/2 system.

We reach the spin 3/2 level. The free spin 1 - spin 3/2 system has the following action and rigid susy transformation rules

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\rho\sigma}\partial_\rho\psi_\sigma, \\ \delta_\epsilon A_\mu &= \bar{\epsilon}\psi_\mu, \quad \delta_\epsilon\psi_\mu = -\frac{1}{2}F_{\mu\nu}\gamma^\nu\epsilon - \frac{i}{2}\tilde{F}_{\mu\nu}\gamma^\nu\gamma_5\epsilon, \end{aligned} \quad (11)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ with $\varepsilon^{0123} = 1$. The fields ψ_μ are Majorana spinors. To prove the susy invariance of the action one needs the following identities $\varepsilon^{\sigma\tau\rho\rho}\varepsilon_{\sigma\tau\alpha\beta} = -2(\delta_\alpha^\mu\delta_\beta^\rho - \delta_\beta^\mu\delta_\alpha^\rho)$, $\gamma^{\mu\rho\sigma\tau} = i\varepsilon^{\mu\rho\sigma\tau}\gamma_5$ and $\gamma^{\rho\sigma}\gamma_5 = -\frac{i}{2}\varepsilon^{\rho\sigma\alpha\beta}\gamma_{\alpha\beta}$ [22]. The Euler-Lagrange variation of the action yields the bosonic boundary term

⁵On a spacelike boundary at fixed time, the curvatures F_{jk} and F_{j0} vanish for these BC, respectively, justifying the names magnetic and electric. On other boundaries we use the same terminology.

$-\delta A_{\mathbf{j}} F^{\mathbf{jn}}$ we discussed above, and the spin 3/2 boundary term $-\bar{\psi}_{\mathbf{i}} \gamma^{\mathbf{ijn}} \delta \psi_{\mathbf{j}}$. (Since the index ρ in $\gamma^{\mu\rho\sigma}$ is along the normal, the other indices $\mu = \mathbf{i}$ and $\sigma = \mathbf{j}$ lie in the boundary). As in the case of spin 1/2 there is no useful boundary term for spin 3/2 as $\bar{\psi}_{\mathbf{i}} \gamma^{\mathbf{ijn}} \psi_{\mathbf{j}}$ vanishes.

Again we need BC on $\psi_{\mathbf{j}}$ without $\partial_{\mathbf{n}}$ derivatives. It is clear that both for $P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ and $P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ this boundary term cancels. Since this time $\delta_{\epsilon} A_{\mathbf{i}} = \bar{\epsilon} \psi_{\mathbf{i}}$ instead of $\bar{\epsilon} \gamma_{\mathbf{i}} \lambda$, the projection operators on λ and $\psi_{\mathbf{i}}$ must be opposite in order that no susy breaking boundary terms occur. Similarly, $P_{\pm} \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ requires $\delta_{\epsilon} P_{\pm} \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$. With the expression for $\delta_{\epsilon} \psi_{\mathbf{j}}$ given above, one finds the following conditions

$$\begin{aligned} P_+ \delta_{\epsilon} \psi_{\mathbf{j}}|_{\partial\mathcal{M}} &= -\frac{1}{2} F_{\mathbf{jn}} \gamma^{\mathbf{n}} (P_+ \epsilon)|_{\partial\mathcal{M}} - \frac{i}{2} \tilde{F}_{\mathbf{jk}} \gamma^{\mathbf{k}} \gamma_5 (P_+ \epsilon)|_{\partial\mathcal{M}}, \\ P_- \delta_{\epsilon} \psi_{\mathbf{j}}|_{\partial\mathcal{M}} &= -\frac{1}{2} F_{\mathbf{jk}} \gamma^{\mathbf{k}} (P_+ \epsilon)|_{\partial\mathcal{M}} - \frac{i}{2} \tilde{F}_{\mathbf{jn}} \gamma^{\mathbf{n}} \gamma_5 (P_+ \epsilon)|_{\partial\mathcal{M}}. \end{aligned} \quad (12)$$

Thus $F_{\mathbf{jn}}|_{\partial\mathcal{M}} = 0$ if $P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$, or $F_{\mathbf{jk}}|_{\partial\mathcal{M}} = 0$ if $P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$. We recall that we split $F^{\mathbf{nj}}|_{\partial\mathcal{M}} = 0$ into $\partial_{\mathbf{n}} A_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ and $A_{\mathbf{n}}|_{\partial\mathcal{M}} = 0$. Our two sets of BC increase as follows

$$\begin{aligned} S|_{\partial\mathcal{M}} &= 0, & \partial^{\mathbf{n}} P|_{\partial\mathcal{M}} &= 0, & P_- \lambda|_{\partial\mathcal{M}} &= 0, \\ A_{\mathbf{n}}|_{\partial\mathcal{M}} &= 0, & \partial_{\mathbf{n}} A_{\mathbf{j}}|_{\partial\mathcal{M}} &= 0, & P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} &= 0, \end{aligned} \quad (13)$$

or

$$\begin{aligned} P|_{\partial\mathcal{M}} &= 0, & \partial^{\mathbf{n}} S|_{\partial\mathcal{M}} &= 0, & P_+ \lambda|_{\partial\mathcal{M}} &= 0, \\ A_{\mathbf{j}}|_{\partial\mathcal{M}} &= 0, & P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} &= 0. \end{aligned} \quad (14)$$

Susy variations yield further conditions. Consider first the set (13) and the susy variation $\delta_{\epsilon} F^{\mathbf{nj}}|_{\partial\mathcal{M}} = \bar{\epsilon} (\partial^{\mathbf{n}} \psi^{\mathbf{j}} - \partial^{\mathbf{j}} \psi^{\mathbf{n}})|_{\partial\mathcal{M}} = \bar{\epsilon} (\partial^{\mathbf{n}} P_- \psi^{\mathbf{j}} - \partial^{\mathbf{j}} P_- \psi^{\mathbf{n}})|_{\partial\mathcal{M}}$. For consistency this expression should vanish. Since we consider local BC only, we like to avoid nonlocal relations between $\psi^{\mathbf{j}}$ and $\psi^{\mathbf{n}}$ on the boundary. Therefore, the two terms in the brackets above should vanish separately,

$$P_- \partial^{\mathbf{n}} \psi^{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad P_- \psi^{\mathbf{n}}|_{\partial\mathcal{M}} = 0. \quad (15)$$

In the second set (14) we find the chain $P_- \partial_{\mathbf{n}} \lambda|_{\partial\mathcal{M}} = 0$, $\partial_{\mathbf{n}} F^{\mathbf{in}}|_{\partial\mathcal{M}} = 0$ which implies $\partial^{\mathbf{n}} A_{\mathbf{n}}|_{\partial\mathcal{M}} = 0$ (and $\partial_{\mathbf{n}}^2 A_{\mathbf{i}}|_{\partial\mathcal{M}} = 0$), and finally $P_- \partial^{\mathbf{n}} \psi_{\mathbf{n}}|_{\partial\mathcal{M}} = 0$. So we recognize a pattern: a given bosonic field has BC with an odd number

of normal indices in one set while in the other set it has BC with an even number of normal indices.

Now we are ready to analyze the free spin 3/2 - spin 2 system. Let us consider small fluctuations of the metric about flat Minkowski background, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. We expand the action of $N = 1$ supergravity in arbitrary dimensions, keeping at most terms quadratic in fluctuations of $h_{\mu\nu}$ and in the gravitino ψ_μ

$$\mathcal{L}^{\text{lin}} = \mathcal{L}_{EH}^{\text{lin}} + \mathcal{L}_\psi^{\text{lin}}. \quad (16)$$

The linearized Einstein-Hilbert gravity action reads through second order in $h_{\mu\nu}$

$$\mathcal{L}_{EH}^{\text{lin}} = \frac{1}{2\kappa}(h^\mu{}_{,\mu} - h_{,\mu}{}^\mu) + \frac{1}{2}\partial_\lambda\Omega^\lambda + \frac{1}{2}\tilde{\mathcal{L}}^{(2)} \quad (17)$$

$$\Omega_\lambda = h^{\mu\nu}\partial_\lambda h_{\mu\nu} - h_{\lambda\nu}(h^\nu{}_{,\mu} - h_{,\mu}{}^\nu) - h^{\mu\nu}\partial_\mu h_{\lambda\nu} \quad (18)$$

$$\tilde{\mathcal{L}}^{(2)} = -\frac{1}{4}(\partial_\mu h_{\nu\sigma})^2 + \frac{1}{2}(\partial_\lambda h_{\mu\nu})(\partial^\mu h^{\lambda\nu}) - \frac{1}{2}h^\mu h_{,\mu} + \frac{1}{4}h^2_{,\mu} \quad (19)$$

Here $h_\mu \equiv \partial^\nu h_{\mu\nu}$, a comma denotes partial differentiation, and $h \equiv h^\mu{}_\mu = h_{\mu\nu}\eta^{\mu\nu}$.

Of course one can always change $\tilde{\mathcal{L}}^{(2)}$ by partial integration, yielding corresponding changes in Ω_λ . The expression for $\tilde{\mathcal{L}}^{(2)}$ in (19) corresponds to the terms quadratic in $h_{\mu\nu}$ one gets from minus the two $\Gamma\Gamma$ terms in the Einstein-Hilbert action. One often uses these latter two terms as action when one studies canonical quantization or gravitational radiation, because in this form there are no double derivatives of $h_{\mu\nu}$ in the action [23]. Note, however, that $\tilde{\mathcal{L}}^{(2)}$ is not the Fierz-Pauli action for free spin 2 fields

$$\mathcal{L}_{\text{FP}}^{(2)} = -\frac{1}{4}(\partial_\mu h_{\nu\sigma})^2 + \frac{1}{2}(h_\mu)^2 - \frac{1}{2}h_\mu h_{,\mu} + \frac{1}{4}h^2_{,\mu}. \quad (20)$$

If one replaces $\tilde{\mathcal{L}}^{(2)}$ by $\mathcal{L}_{\text{FP}}^{(2)}$ one finds that Ω_λ is replaced by $\Omega_{\lambda(\text{FP})} = h_{\mu\nu}\partial_\lambda h^{\mu\nu} - h_{\lambda\nu}(\frac{3}{2}h^\nu{}_{,\mu} - h_{,\mu}{}^\nu)$. We shall continue with (17) - (19).

The linearized gravitino action was given above

$$\mathcal{L}_\psi^{\text{lin}} = -\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\rho\sigma}\partial_\rho\psi_\sigma \quad (21)$$

where for example $\gamma^{123} = \gamma^1\gamma^2\gamma^3$. Possible boundary terms which may be added to the action (17) will be discussed later for each set of BC separately.

If one neglects all boundary terms the action (17) is invariant under the following rigid susy transformations

$$\begin{aligned}\delta_\epsilon h_{\mu\nu} &= \frac{1}{2} (\bar{\epsilon} \gamma_\mu \psi_\nu + \bar{\epsilon} \gamma_\nu \psi_\mu), \quad \delta_\epsilon \psi_\mu = \frac{1}{4} (\omega_\mu{}^{mn})_{\text{lin}} \gamma_{mn} \epsilon \\ (\omega_{\mu mn})_{\text{lin}} &= \frac{1}{2} (\partial_n h_{\mu m} - \partial_m h_{\mu n})\end{aligned}\quad (22)$$

This expression for the linearized spin connection⁶ is easily obtained from (89), using $e_{\mu m} = \delta_{\mu m} + \frac{\kappa}{2} h_{\mu m} + \mathcal{O}(h^2)$.

We would like to extend the two sets of BC (13) and (14) to the gravitational field $h_{\mu\nu}$. Consider (13) first. We start with the requirement that the orbit of BC must be closed under the rigid susy transformations (22). The susy variation of $h_{\mathbf{j}\mathbf{k}}$ on the boundary reads

$$\delta_\epsilon h_{\mathbf{j}\mathbf{k}}|_{\partial\mathcal{M}} = \frac{1}{2} (\bar{\epsilon} \gamma_{\mathbf{j}} P_- \psi_{\mathbf{k}} + \bar{\epsilon} \gamma_{\mathbf{k}} P_- \psi_{\mathbf{j}})|_{\partial\mathcal{M}} = \frac{1}{2} (\bar{\epsilon} P_+ \gamma_{\mathbf{j}} \psi_{\mathbf{k}} + \bar{\epsilon} P_+ \gamma_{\mathbf{k}} \psi_{\mathbf{j}})|_{\partial\mathcal{M}} = 0 \quad (23)$$

where we used (13) and (5). This equation allows us to impose the BC $h_{\mathbf{j}\mathbf{k}}|_{\partial\mathcal{M}} = 0$ where we immediately recognize a linearized version of the Gibbons-Hawking boundary condition (2). Therefore, the first of our two sets of BC reads

$$\begin{aligned}S|_{\partial\mathcal{M}} &= 0, \quad \partial^{\mathbf{n}} P|_{\partial\mathcal{M}} = 0, \quad P_- \lambda|_{\partial\mathcal{M}} = 0, \\ A_{\mathbf{n}}|_{\partial\mathcal{M}} &= 0, \quad \partial_{\mathbf{n}} A_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad h_{\mathbf{j}\mathbf{k}}|_{\partial\mathcal{M}} = 0.\end{aligned}\quad (24)$$

This should be consistent with the BC $P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$, whose susy variation yield $(\omega_{\mathbf{j}\mathbf{k}\mathbf{l}})_{\text{lin}}|_{\partial\mathcal{M}} = 0$. This is indeed consistent with $h_{\mathbf{j}\mathbf{k}}|_{\partial\mathcal{M}} = 0$. Consistency of the BC (15) for $\psi_{\mathbf{n}}$ yields $\partial_{\mathbf{n}} h_{\mathbf{i}\mathbf{n}}|_{\partial\mathcal{M}} = 0$ and $h_{\mathbf{n}\mathbf{n}}|_{\partial\mathcal{M}} = 0$, all with an even number of \mathbf{n} indices.

Let us now now discuss which boundary terms should be added to the action (17) to make this set of BC fully consistent. The Euler-Lagrange variation of (17) yields a boundary term already at the *linear* order

$$-\frac{1}{2\kappa} \int_{\partial\mathcal{M}} d^3x \partial_{\mathbf{n}} \delta h_{\mathbf{i}}{}^{\mathbf{i}}, \quad (25)$$

where we neglected the terms with $\partial_{\mathbf{i}} \delta h^{\mathbf{n}\mathbf{i}}$ which are total derivatives on the boundary. The occurrence of terms in the variation which are linear in fields

⁶There is also a term $-\frac{1}{2} \partial_\mu (e_{mn} - e_{nm})$ in $(\omega_{\mu mn})_{\text{lin}}$, but it does not contribute to the susy variation of the action because it is a linearized Lorentz transformation.

distinguishes gravity from other field theories. Since we have already imposed the Dirichlet BC on $h_{\mathbf{ij}}$ we cannot impose also the Neumann BC on the same components $h_{\mathbf{ij}}$. Therefore, one has to add a boundary term in order to cancel (25). The only appropriate boundary invariant is the trace of the extrinsic curvature integrated over the boundary. By comparing this invariant to (25) we fix the coefficient in front of it and arrive at the York-Gibbons-Hawking boundary term⁷

$$\mathcal{S}_{\text{YGH}} = \frac{1}{\kappa^2} \int_{\partial\mathcal{M}} d^3x \sqrt{|\det g_{\mathbf{ij}}|} K_{\mathbf{i}}{}^{\mathbf{i}} \simeq \frac{1}{2\kappa} \int_{\partial\mathcal{M}} d^3x \partial_{\mathbf{n}} \delta h_{\mathbf{i}}{}^{\mathbf{i}} + \mathcal{O}(h^2). \quad (26)$$

As a check one might prove that the boundary terms produced by the susy variation and by the Euler-Lagrange variation of the action (16), supplemented by the terms quadratic in $h_{\mu\nu}$ in the York-Gibbons-Hawking term (26), vanish to next (quadratic) order as well if one uses (24) and $h_{\mathbf{nn}}|_{\partial\mathcal{M}} = h_{\mathbf{in},\mathbf{n}}|_{\partial\mathcal{M}} = 0$. This calculation was done in [6] in the full non-linear theory and we shall not repeat it here. We only note that the BC on $h_{\mathbf{ij}}$, $\partial_{\mathbf{n}} h_{\mathbf{in}}$ and $h_{\mathbf{nn}}$ require that $\xi^{\mathbf{n}}|_{\partial\mathcal{M}} = \partial_{\mathbf{n}} \xi^{\mathbf{i}}|_{\partial\mathcal{M}} = 0$. The authors of [6] introduced a fermionic boundary term which vanishes under the BC on the gravitino and therefore does not affect the proof. The paper [6] did not obtain an orbit of BC closed under the local susy transformations. (We shall show in section 6 that such an orbit with the Gibbons-Hawking BC on the metric fluctuations does not exist for local BC).

Let us now turn to the other set of BC (14). Since $P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$, for consistency we also request $P_- \delta_\epsilon \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$. This condition yields

$$0 = P_- \delta_\epsilon \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = \frac{1}{4} (\partial_{\mathbf{k}} h_{\mathbf{n}\mathbf{j}} - \partial_{\mathbf{n}} h_{\mathbf{k}\mathbf{j}}) \gamma^{\mathbf{nk}} P_+ \epsilon|_{\partial\mathcal{M}}. \quad (27)$$

Therefore, we require

$$h_{\mathbf{n}\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad \partial_{\mathbf{n}} h_{\mathbf{k}\mathbf{j}}|_{\partial\mathcal{M}} = 0. \quad (28)$$

⁷Since the BC $h_{\mathbf{jk}}|_{\partial\mathcal{M}} = 0$ is not preserved by general coordinate transformations with $\xi^{\mathbf{j}}$, the argument based on 3 covariance of the boundary term is not totally convincing. However, in the quadratic order one can easily classify all possible boundary terms containing two fields $h_{\mu\nu}$ and one derivative which can be added to the York-Gibbons-Hawking term. All such boundary terms either vanish identically due to the BC which are already imposed, or their Euler-Lagrange variations produce additional BC which overconstrain the system.

Now (25) vanishes. Next we compare (28) with the expression in (121) for the extrinsic curvature. In the linearized case (28) implies that the extrinsic curvature vanishes

$$K_{\mathbf{j}\mathbf{k}}|_{\partial\mathcal{M}} = 0 \quad (29)$$

Due to this BC the only boundary term we can add, namely the extrinsic curvature, vanishes together with its susy variations. Our second set of BC increases to

$$\begin{aligned} P|_{\partial\mathcal{M}} = 0, \quad \partial^{\mathbf{n}}S|_{\partial\mathcal{M}} = 0, \quad P_+\lambda|_{\partial\mathcal{M}} = 0, \\ A_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad P_-\psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad h_{\mathbf{n}\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad \partial_{\mathbf{n}}h_{\mathbf{i}\mathbf{j}}|_{\partial\mathcal{M}} = 0. \end{aligned} \quad (30)$$

By closing this set with respect to other symmetry transformations one arrives at further BC which we leave to the reader to derive. They repeat the patterns we found before. One can also obtain the remaining boundary conditions by taking the linearized limit of the BC in full non-linear supergravity (see sec. 6). This shows that the set (30) should be supplemented by $\partial_{\mathbf{n}}h_{\mathbf{nn}}|_{\partial\mathcal{M}} = 0$ in agreement with our rules for the number of normal indices. In particular we find $h_{\mathbf{n}}|_{\partial\mathcal{M}} = h_{,\mathbf{n}}|_{\partial\mathcal{M}} = 0$ in the second sector. By using this property and (30), it is easy to show that $\Omega_{\mathbf{n}}$ vanishes on the boundary and that one can integrate by parts in $\tilde{\mathcal{L}}^{(2)}$ without creating boundary terms. For the same reason one cannot write a non-zero boundary term for $h_{\mu\nu}$. Indeed, any relevant boundary term has the mass dimension one, i.e. it contains a single derivatives and, consequently, an odd total number of vector indices. Since all tangential (boundary) indices must be contracted in pairs, one has an odd number of normal indices. All such terms vanish on the boundary.

3 Backgrounds with rigid susy

We next study consistent BC in a model with rigid susy and a boundary term: the susy kink in 1+1 dimensions with the kink soliton $\phi_K(x)$ as background. The action reads

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}F^2 - \frac{1}{2}\bar{\psi}\not{\partial}\psi - \frac{1}{2}U'\bar{\psi}\psi + FU \quad (31)$$

$$U(\phi) = \sqrt{\frac{g}{2}}\left(\phi^2 - \frac{\mu^2}{2g}\right); \quad \phi = \phi_K(x) + \eta(x, t); \quad \partial_x\phi_K + U(\phi_K) = 0 \quad (32)$$

All fields are real. The susy transformation rules for η , F and ψ with rigid parameter ϵ read, using

$$\begin{aligned}\gamma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} \\ \delta\phi &= \bar{\epsilon}\psi, \\ \delta\psi &= \gamma^\mu \partial_\mu \phi \epsilon + F\epsilon, \\ \delta F &= \bar{\epsilon}\gamma^\mu \partial_\mu \psi.\end{aligned}\tag{33}$$

Eliminating the auxiliary field F by $F = -U$ yields a term $-\frac{1}{2}U^2$ in the action, while the transformation rules become

$$\begin{aligned}\delta\psi_+ &= \partial_x \phi \epsilon_+ - \partial_t \phi \epsilon_- - U\epsilon_+, \\ \delta\psi_- &= -\partial_x \phi \epsilon_- + \partial_t \phi \epsilon_+ - U\epsilon_-, \\ \delta\eta &= -i\epsilon_+ \psi_- + i\epsilon_- \psi_+.\end{aligned}\tag{34}$$

The kink background $\phi = \phi_K$, $\psi = 0$ satisfies the field equation $\partial_x \phi_K + U(\phi_K) = 0$, and this background is clearly invariant under susy transformations with ϵ_- . So all terms in (34) are at least linear in quantum fluctuations. Let us now study BC in this model. We consider a boundary in space at fixed x^1 .

From the field equations one finds the boundary term

$$\int_{-\infty}^{\infty} \left[-\delta\phi \partial_x \phi - \frac{1}{2} \bar{\psi} \gamma^1 \delta\psi \right] dt\tag{35}$$

For ψ one finds as before $P_\pm \psi = 0$, which becomes with $P_\pm = \frac{1}{2}(1 \pm \gamma^1)$ just $\psi_\pm = 0$. However, as it stands the model cannot implement the Neumann BC $\partial_x \phi = 0$ because the background does not satisfy this condition. To remedy this, one can add a boundary term $\int K(\phi) dt$ [2]. Then one finds the following BC: $\delta\eta(-\partial_x \phi + K') = 0$. Therefore, either $\eta|_{\partial\mathcal{M}} = 0$ or $(-\partial_x \phi + K')|_{\partial\mathcal{M}} = 0$. In order that the second BC (modified Neumann) holds to zeroth order in η one finds $K' = -U$. For the Dirichlet BC, of course, no restrictions on K follow but since $K = K(\phi_K)$ in that case, one may omit this boundary action altogether. To linear order in η one finds from the field equations the following BC for the fluctuations

$$\eta|_{\partial\mathcal{M}} = 0 \quad \text{or} \quad (\partial_x + U')\eta = 0, \quad \text{and} \quad \psi_+ = 0 \quad \text{or} \quad \psi_- = 0.\tag{36}$$

Rigid susy is preserved provided the boundary terms generated by an ϵ_- susy variation cancel. One finds then to linear order in fluctuations two sets of BC⁸ which form a subset of (36)

$$\{\eta|_{\partial\mathcal{M}} = 0, \quad \psi_+|_{\partial\mathcal{M}} = 0\} \quad \text{or} \quad \{(\partial_x + U')\eta|_{\partial\mathcal{M}} = 0, \quad \psi_-|_{\partial\mathcal{M}} = 0\} \quad (37)$$

The first set is closed under susy, but the second set leads to a further BC

$$\delta(\epsilon_-)(\partial_x + U')\eta|_{\partial\mathcal{M}} = (\partial_x + U')\psi_+|_{\partial\mathcal{M}} = 0 \quad (38)$$

The latter BC, $(\partial_x + U')\psi_+|_{\partial\mathcal{M}} = 0$, transforms back into $(\partial_x + U')\eta|_{\partial\mathcal{M}}$. So $\eta = 0$ and $\psi_+ = 0$ form a closed system, as do of course $(\partial_x + U')\eta$ and $(\partial_x + U')\psi_+ = 0$, but we also found the BC $\psi_-|_{\partial\mathcal{M}} = 0$ in the second set, for reasons we now explain.

In general one expects to need BC with $P_{\pm}\lambda = 0$ and $P_{\mp}(\partial_n\lambda + \dots) = 0$ for fermions, and *either* $\eta = 0$ or $(\partial_n\eta + \dots) = 0$ for bosons. We saw this happening in the action (3), but due to the nontrivial background the Neumann conditions have now acquired extra terms. The second set of BC indeed has this form, but the first set misses a BC with $\partial_x\psi_-$. The nontrivial soliton background has already eliminated half of the susy, and this seems to be the reason for the unexpected form of the first set of BC. In ref [24] it was shown that if one imposes field equations one finds in the first set the “missing BC”

$$(\partial_x - U')\psi_-|_{\partial\mathcal{M}} = 0, \quad (39)$$

but off-shell our approach does not lead to (39) as a BC.

4 Quantum mechanical supergravity

As a warming up exercise for theories with local susy we now consider a simple model for supergravity where all nonlinearities can easily be dealt with: the quantum mechanics of a bosonic point particle $\varphi(t)$ and a one-component fermionic point particle $\lambda(t)$ coupled to an external one-component gravitational field $h(t)$ and to an external one-component gravitino field $\psi(t)$. All fields are real. (There do not, of course, exist gauge actions for h and ψ in one dimension). This model is known to be locally supersymmetric [25], and

⁸To higher order in fluctuations one finds no new BC, but rather these BC are modified by terms of higher order in fluctuating fields.

its BRST properties, both for the Lagrangian and the Hamiltonian formulation, as well as its superspace formulation have recently been worked out in [26]. In none of these articles boundary terms have been discussed; that is the subject of this section. An interesting aspect of this analysis is that one obtains boundary conditions in time. In the next section we consider the Hamiltonian formulation of this model and discuss possible boundary actions.

The classical action reads

$$L = \frac{1}{2}(1 - 2h)\dot{\varphi}^2 + \frac{i}{2}(1 - 2h)\lambda\dot{\lambda} - i\psi\dot{\varphi}\lambda \quad (40)$$

and is invariant under the following reparametrization and local susy transformations

$$\delta\varphi = \xi\dot{\varphi} + i\epsilon(1 - 2h)\lambda, \quad (41)$$

$$\delta\lambda = \xi\dot{\lambda} + \frac{1}{2}\dot{\xi}\lambda - (1 - 2h)\dot{\varphi}\epsilon, \quad (42)$$

$$\delta h = \frac{1}{2}\dot{\xi} + \xi\dot{h} - \dot{\xi}h - (1 - 2h)i\epsilon\psi, \quad (43)$$

$$\delta\psi = \xi\dot{\psi} - \frac{1}{2}\dot{\xi}\psi + (1 - 2h)\left[(1 - 2h)\dot{\epsilon} + \dot{h}\epsilon\right] \quad (44)$$

In this model the susy Noether current $\dot{\varphi}\lambda$ in flat space varies into the current $-(\dot{\varphi}\dot{\varphi} + i\lambda\dot{\lambda})\epsilon$ which couples to h . This is *not* the model one gets by putting the Dirac action $\frac{i}{2}\lambda\dot{\lambda}$ in curved space, because even in curved space the Dirac action in one dimension remains $\frac{i}{2}\lambda\dot{\lambda}$, without h field. However, one can rescale λ , ψ and ϵ , and then one finds the action⁹ without coupling $ih\lambda\dot{\lambda}$. We shall use the latter model for our BRST analysis, but continue for the time being with the former model; readers may of course interchange the models for either analysis.

The action is obtained by integrating L over a finite time interval, and we study the BC at one of the two endpoints¹⁰. The field equations lead to

⁹Both models are special cases of a one-parameter class of actions which are obtained from the Noether method. The rescalings are $\psi = (1 + 2hx)^{1/2}\hat{\psi}$, $\epsilon = (1 + 2hx)^{1/2}\hat{\epsilon}$ and $(1 + 2hx)^{1/2}\lambda = \hat{\lambda}$. For $x = 0$ and $x = -1$ one finds actions in polynomial form, and $x = 0$ gives (40), while $x = -1$ yields the action (54) [26].

¹⁰This corresponds to open string theory, but with BC in time, which have never been worked out in string theory as far as we know. For open strings one has two ghosts and antighosts (c^+ , c^- and b_{++} , b_{--}) and then the BC become $c^+ = c^-$ and $b_{++} = b_{--}$ instead of $c = 0$ and $b = 0$.

the boundary terms

$$\delta\varphi [(1-2h)\dot{\varphi} - i\psi\lambda] + \frac{i}{2}(1-2h)\lambda\delta\lambda. \quad (45)$$

These vanish provided¹¹

$$\lambda|_{\partial\mathcal{M}} = 0 \quad (46)$$

and

$$\varphi|_{\partial\mathcal{M}} = 0 \quad (\text{Dirichlet}) \quad (47)$$

or

$$\dot{\varphi}|_{\partial\mathcal{M}} = \partial_{\mathbf{n}}\varphi|_{\partial\mathcal{M}} = 0 \quad (\text{Neumann}). \quad (48)$$

For a one-component fermion λ one cannot, of course, define P_+ or P_- .

From a general coordinate transformation one finds $\delta L = \frac{d}{dt}(\xi L)$ (as usual, and one easily checks), so ξ vanishes at the boundary, $\xi|_{\partial\mathcal{M}} = 0$. Making a local susy transformation yields the usual boundary terms by partially integrating the kinetic terms of φ and λ . This yields the following boundary terms

$$\dot{\varphi}i\epsilon(1-2h)\lambda - \frac{i}{2}(1-2h)^2\dot{\varphi}\epsilon\lambda. \quad (49)$$

These terms vanish since we already know that $\lambda = 0$ at the boundary. So the action is Einstein (general coordinate) and local susy invariant, but there is of course no local Lorentz invariance to be dealt with in this model.

Consistency of $\lambda = 0$ and either $\varphi = 0$ or $\dot{\varphi} = 0$ requires that also symmetry transformations of these constraints vanish at the boundary. This is the case for ξ transformations if ξ vanish at the boundary

$$\xi|_{\partial\mathcal{M}} = 0. \quad (50)$$

For example $\delta_\xi\dot{\varphi} = \xi\partial_t^2\varphi + \dot{\xi}\dot{\varphi} = 0$ when $\xi = 0$ and $\dot{\varphi} = 0$. For local susy we find from $\lambda = 0$

$$0 = \delta_\epsilon\lambda|_{\partial\mathcal{M}} = (1-2h)\dot{\varphi}\epsilon|_{\partial\mathcal{M}} \quad (51)$$

Consequently, $\epsilon|_{\partial\mathcal{M}} = 0$ if $\varphi|_{\partial\mathcal{M}} = 0$. If $\varphi|_{\partial\mathcal{M}} = 0$ we obtain from $\delta_\epsilon\varphi = i\epsilon(1-2h)\lambda$ so we find no new BC, but if $\dot{\varphi} = 0$ we find from $\delta_\epsilon\dot{\varphi}|_{\partial\mathcal{M}} = 0$ that also in this case ϵ vanishes at the boundary

$$\epsilon|_{\partial\mathcal{M}} = 0. \quad (52)$$

¹¹Another solution contains $h = 1/2$; since in this case the whole action collapses, we do not consider this case.

There are no auxiliary fields needed in this model, and the local gauge algebra indeed closes. One finds

$$[\delta_s(\epsilon_2), \delta_s(\epsilon_1)] = \delta_s(\tilde{\epsilon} = -2i\epsilon_2\epsilon_1\psi) + \delta_g(\tilde{\xi} = (1-2h)2i\epsilon_2\epsilon_1). \quad (53)$$

Consistency requires that $\tilde{\xi}$ and $\tilde{\epsilon}$ vanish at the boundary as they clearly do. Other commutators read

$$\begin{aligned} [\delta_s(\epsilon), \delta_g(\xi)] &= \delta_s(\tilde{\epsilon} = \xi\dot{\epsilon}), \\ [\delta_g(\xi_2), \delta_g(\xi_1)] &= \delta_g(\tilde{\xi} = \xi_1\dot{\xi}_2 - \xi_2\dot{\xi}_1) \end{aligned}$$

and also this time $\tilde{\xi}$ and $\tilde{\epsilon}$ vanish at the boundary.

Consider next the BRST symmetry. We start from the classical action¹²

$$L_{\text{cl}} = \frac{1}{2}\dot{\varphi}^2 + \frac{i}{2}\lambda\dot{\lambda} - h\dot{\varphi}^2 - i\psi\dot{\varphi}\lambda \quad (54)$$

and add the nonderivative gauge fixing term one uses in string theory

$$L_{\text{fix}} = dh + \Delta\psi, \quad (55)$$

where d and Δ are the BRST auxiliary fields.

The BRST rules are

$$\begin{aligned} \delta_B\varphi &= [\dot{\varphi}c - \lambda\gamma]\Lambda, \\ \delta_B\lambda &= [\dot{\lambda}c + i(1-2h)\dot{\varphi}\gamma + \psi\lambda\gamma]\Lambda, \\ \delta_Bh &= \left[\frac{1}{2}(1-2h)\dot{c} + \dot{h}c + (1-2h)\psi\gamma\right]\Lambda, \\ \delta_B\psi &= [\dot{\psi} - i(1-2h)\dot{\gamma}]\Lambda, \end{aligned} \quad (56)$$

where Λ is the constant anticommuting imaginary BRST parameter, the real c is the coordinate ghost ($\xi = \xi\Lambda$) and the real γ is the susy ghost ($\epsilon = -i\gamma\Lambda$). The ghost action becomes

$$\begin{aligned} L_{\text{ghost}} &= b \left[\frac{1}{2}(1-2h)\dot{c} + \dot{h}c + (1-2h)\psi\gamma \right] \\ &\quad + \beta \left[-i(1-2h)\dot{\gamma} + \dot{\psi}c \right] \end{aligned} \quad (57)$$

¹²We mentioned this action in footnote 9. It is this form of the action which can be straightforwardly written in superspace [26].

where the antihermitian b is the coordinate antighost and the antihermitian β the susy antighost. The transformation rules of the ghosts follow from the closure of the local gauge algebra (or from the invariance of the action)

$$\begin{aligned}\delta_B c &= [-c\dot{c} + i(1 - 2h)\gamma\gamma]\Lambda \\ \delta_B \gamma &= [c\dot{\gamma} + \psi\gamma\gamma]\Lambda\end{aligned}\tag{58}$$

and as usual the antighosts and the auxiliary fields form contactable pairs

$$\begin{aligned}\delta_B b &= \Lambda d, & \delta_B d &= 0 \\ \delta_B \beta &= \Lambda \Delta, & \delta_B \Delta &= 0.\end{aligned}\tag{59}$$

From the field equations we obtain again the BC in (46) - (48), and further

$$(b\delta c)|_{\partial\mathcal{M}} = 0, \quad (\delta h b c)|_{\partial\mathcal{M}} = 0, \quad (\beta\delta\gamma)|_{\partial\mathcal{M}} = 0, \quad (\beta\delta\psi c)|_{\partial\mathcal{M}} = 0.\tag{60}$$

Before solving these, we consider other points on the orbit of BC. Let us check that BRST symmetry preserves the boundary conditions. From $\delta_B \lambda|_{\partial\mathcal{M}} = 0$ one obtains

$$c|_{\partial\mathcal{M}} = 0, \quad (\dot{\varphi}\gamma)|_{\partial\mathcal{M}} = 0\tag{61}$$

From $\delta_B \varphi|_{\partial\mathcal{M}} = 0$ one finds $c|_{\partial\mathcal{M}} = 0$ and $\lambda\gamma|_{\partial\mathcal{M}} = 0$, both of which are satisfied. Since $\dot{\varphi}|_{\partial\mathcal{M}} \neq 0$ in this case, we also find that

$$\gamma|_{\partial\mathcal{M}} = 0\tag{62}$$

On the other hand, $\dot{\varphi}|_{\partial\mathcal{M}} = 0$ implies that $\dot{\lambda}\gamma = 0$, and since $\dot{\lambda}|_{\partial\mathcal{M}} \neq 0$, we conclude that also in this case the susy ghost vanishes at the boundary.

Hence, all ghosts vanish at the boundary. This is due to the algebraic gauge choice $h = \psi = 0$; for de Donder type of gauge for h or a gauge choice $\psi \frac{d}{dt} \psi$ for ψ one would get different results. Also the BC for the (anti)ghosts in (60) are then satisfied. For consistency the BC $c|_{\partial\mathcal{M}} = \gamma|_{\partial\mathcal{M}} = 0$ requires that also $\delta_B c|_{\partial\mathcal{M}} = \delta_B \gamma|_{\partial\mathcal{M}} = 0$. This is indeed the case as one checks from (58), and hence we conclude that the total consistent set of BC consists of

$$\begin{aligned}\lambda|_{\partial\mathcal{M}} &= 0, & \varphi|_{\partial\mathcal{M}} &= 0 \text{ or } \dot{\varphi}|_{\partial\mathcal{M}} = 0, \\ c|_{\partial\mathcal{M}} &= 0, & \gamma|_{\partial\mathcal{M}} &= 0, & \xi|_{\partial\mathcal{M}} &= 0, & \epsilon|_{\partial\mathcal{M}} &= 0\end{aligned}\tag{63}$$

There are no BC on h, ψ, b, β in this quantum mechanical model with one-component fields. We consider possible boundary terms in the next section.

5 Hamiltonian boundary conditions

So far we have been discussing models in the Lagrange formalism. In the Hamiltonian formalism the issue of BC is simpler because the action is of the form $L = \dot{Q}P - H$ where $H = H(Q, P)$ does not contain any derivatives. BC enter then the path integral as conditions on the states at initial and final times [9]. Conditions involving time derivatives, of the form $\partial_{\mathbf{n}}\varphi = 0$, become now conditions on momenta, hence one needs only to specify the values of (half of the) fields and momenta at the boundary. If these states are physical states, they should be annihilated by the BRST charge Q . This raises the question whether $Q = 0$ is equivalent to the BC one gets from our program. Our BC are, of course, off-shell, whereas the ones from $Q = 0$ are on-shell. We use the quantum mechanical model (54) again to study these issues in a concrete way.

The action in Hamiltonian form is given by

$$L = \dot{\varphi}p + \dot{\lambda}\pi_{\lambda} + \dot{G}p_G + \dot{\Psi}\pi_{\Psi} + \dot{c}\pi_c + \dot{b}p_b + \dot{\gamma}p_{\gamma} + \dot{\beta}\pi_{\beta} + \{Q_H, \psi_g\}. \quad (64)$$

where Q_H is the nilpotent quantum BRST charge in the Hamiltonian formalism, which has in our case the form

$$Q_H = \frac{1}{2}cp^2 - i\gamma p(\pi_{\lambda} - \frac{i}{2}\lambda) + p_b p_G + \pi_{\beta}\pi_{\Psi} - i\pi_c\gamma\gamma, \quad (65)$$

In a general Hamiltonian framework the action has the form

$$L = \dot{q}^i p_i - H + \{Q_H, \psi_g\}, \quad (66)$$

but the quantum Hamiltonian H which commutes with Q_H vanishes in our case. The transformation rules which leave the classical action invariant up to boundary terms are as follows. The diffeomorphisms are generated by $\frac{1}{2}p^2$. This yields

$$\begin{aligned} \delta\varphi &= \hat{\xi}p, & \delta p &= 0, & \delta\lambda &= 0, & \delta\pi_{\lambda} &= 0, \\ \delta\Psi &= 0, & \delta(1 + 2H) &= \frac{d}{dt}\hat{\xi}, & \hat{\xi} &= (1 + 2H)\xi. \end{aligned} \quad (67)$$

The classical gauge fields are $G = (1 + 2H)$ and Ψ and they transform in general as

$$\delta h^A = \frac{d}{dt}\epsilon^A + f_{BC}^A h_{\mu}^B \epsilon^C, \quad (68)$$

where f_{BC}^A are the structure functions of the local gauge algebra. The local susy transformations are generated by $ip(\pi_\lambda - \frac{i}{2}\lambda)$. One finds using Dirac brackets,

$$\begin{aligned}\delta\varphi &= -\epsilon(\pi_\lambda - \frac{i}{2}\lambda), & \delta p &= 0, & \delta\lambda &= -p\epsilon, & \delta\pi_\lambda &= \frac{i}{2}p\epsilon. \\ \delta\Psi &= \dot{\epsilon}, & \delta(1+2H) &= -2i\epsilon\Psi.\end{aligned}\quad (69)$$

We choose as gauge fermion

$$\psi_g = -iG\pi_c - \psi p_\gamma \quad (70)$$

and find for the gauge-artefacts

$$\{Q_H, \psi_g\} = -\frac{1}{2}Gp^2 + \Psi p(p_\lambda - \frac{i}{2}\lambda) + 2\Psi\pi_c\gamma + \pi_cp_b + \pi_\beta ip_\gamma \quad (71)$$

Eliminating π_c , p_b , p_γ and π_b yields

$$\dot{c} = p_b, \quad \dot{\gamma} = -i\pi_\beta \quad (72)$$

and inserting these result back into the action yields

$$\begin{aligned}L &= \dot{\varphi}p + \dot{\lambda}\pi_\lambda + \dot{G}p_G + \dot{\Psi}\pi_\Psi \\ &\quad - \frac{1}{2}Gp^2 + \Psi p(\pi_\lambda - \frac{i}{2}\lambda) + \dot{b}(\dot{c} + 2\gamma\Psi) + \dot{\beta}(i\dot{\gamma})\end{aligned}\quad (73)$$

The first line contains the kinetic terms and the gauge fixing terms $\dot{G} = \dot{\Psi} = 0$, the second line contains the two first class constraints and the ghost actions. (The gauge fixing fermion with G and ψ in the Hamiltonian approach has led to gauge fixing term with \dot{G} and $\dot{\psi}$ in the Lagrangian approach. One can also get G and ψ in the Lagrangian approach if one takes singular limits).

The BC which follow from the field equations are

$$\begin{aligned}\varphi &= 0 \quad \text{or} \quad p = 0, \quad \lambda = 0 \quad \text{or} \quad \pi_\lambda = 0, \quad G = 0 \quad \text{or} \quad p_G = 0, \\ \Psi &= 0 \quad \text{or} \quad \pi_\Psi = 0, \quad b = 0 \quad \text{or} \quad \dot{c} + 2\gamma\Psi = 0, \quad \dot{b} = 0 \quad \text{or} \quad c = 0, \\ \beta &= 0 \quad \text{or} \quad \dot{\gamma} = 0, \quad \dot{\beta} = 0 \quad \text{or} \quad \gamma = 0\end{aligned}\quad (74)$$

The BRST transformation rules for this model read

$$\begin{aligned}\delta_B\varphi &= cp\Lambda - i(\pi_\lambda - \frac{i}{2}\lambda)\gamma\Lambda, & \delta_Bp &= 0, \\ \delta_B\lambda &= i\gamma p\Lambda, & \delta_B\pi_\lambda &= \frac{1}{2}\gamma p\Lambda, \\ \delta_BG &= p_b\Lambda, & \delta_B\Psi &= -\pi_\beta\Lambda, \\ \delta_Bc &= i\gamma\gamma\Lambda, & \delta_Bb &= -\Lambda p_G, \\ \delta_B\gamma &= 0, & \delta_B\beta &= \pi_\psi\Lambda, \\ \delta_B\pi_c &= -\frac{1}{2}p^2\Lambda, & \delta_Bp_\gamma &= 2i\pi_c\gamma\Lambda + ip(\pi_\lambda - \frac{i}{2}\lambda). \\ \delta_Bp_G &= \delta_B\pi_\Psi = \delta_Bp_b = \delta_B\pi_\beta = 0,\end{aligned}\quad (75)$$

Then the boundary terms which should vanish if BRST symmetry is to be exact, are given by

$$cp^2 - i(\pi_\lambda - \frac{i}{2}\lambda)\gamma p - i\gamma p\pi_\lambda + p_b p_G + \pi_\beta \pi_\psi \quad (76)$$

There are many solutions. One consistent set of BC for BRST symmetry is

$$c = 0, \gamma = 0 \quad (\text{corresponding to } \hat{\xi} = 0 \text{ and } \epsilon = 0) \quad (77)$$

$$b = 0, \beta = 0 \quad (\text{since } \dot{c} + 2\gamma\psi = 0 \text{ and } \dot{\gamma} = 0 \text{ are ruled out}) \quad (78)$$

$$p_G = 0, \pi_\Psi = 0 \quad (\text{these are the BRST auxiliary fields}) \quad (79)$$

$$\varphi = 0 \quad \text{or} \quad p = 0 \quad (\text{Dirichlet or Neumann}) \quad (80)$$

$$\lambda = 0 \quad \text{or} \quad \pi_\lambda = 0 \quad (81)$$

In the Hamiltonian formalism, $\pi_\lambda + \frac{i}{2}\lambda = 0$ is a second class constraint, so π_λ transforms like $-\frac{i}{2}\lambda$, and one may therefore replace π_λ by $-\frac{i}{2}\lambda$. Then there is only one BC on λ , namely $\lambda = 0$.

Consistency requires now that also the BRST variation of these invariants vanish. This is the case.

The BRST charge in (65) vanishes provided

$$c = 0, \gamma = 0, p_G = 0, \pi_\Psi = 0 \quad (82)$$

This only a subset of our consistent set of BC.

The conclusion is that requiring the BRST charge to vanish at initial or final times leads only to a subset of all BC needed for consistency as we have defined it. Probably, in addition to the vanishing BRST charge, one should also claim that the symplectic structure is well defined (see [27] and earlier papers [28]). A general discussion of BC in the Hamiltonian formulation of gravity theories can be found in [29].

One can also apply the framework of the Hamiltonian approach to boundaries in a space-like direction. In this case the BC become Hamiltonian constraints and modify the Dirac brackets between boundary values of the fields. A general framework for this procedure was developed in the papers [30] where one can also find further references. More recently this approach was applied to the Dirichlet branes [31].

6 $N = 1$ supergravity in $2 + 1$ dimensions

In this section we present an example of a complete set of consistent boundary conditions for a full nonlinear supergravity model. As model we choose supergravity in $2 + 1$ dimensions which is a bit simpler than supergravity in $3 + 1$ dimensions. (Note that in $1 + 1$ dimensions no gauge action for supergravity exists). The Lagrangian of simple ($N = 1$) supergravity in 3-dimensional Minkowski space reads

$$\mathcal{L} = -\frac{e}{2\kappa^2} R_{\mu\nu}{}^{mn} e_m^\nu e_n^\mu + \frac{1}{2} \bar{\psi}_\mu D_\rho \psi_\sigma \varepsilon^{\mu\rho\sigma} - \frac{e}{2} S^2. \quad (83)$$

where the real scalar S is an auxiliary field and $D_\rho \psi_\sigma = \partial_\rho \psi_\sigma + \frac{1}{4} \omega_\rho^{nm} \gamma_{nm} \psi_\sigma$. Hence

$$\begin{aligned} [D_\mu, D_\nu] &= \frac{1}{4} \gamma_{mn} R_{\mu\nu}{}^{mn}, \\ R_{\mu\nu}{}^{mn} &= \partial_\mu \omega_\nu{}^{mn} - \partial_\nu \omega_\mu{}^{mn} + \omega_\mu{}^m{}_k \omega_\nu{}^{kn} - \omega_\nu{}^m{}_k \omega_\mu{}^{kn}. \end{aligned} \quad (84)$$

Simple counting of the number of field components minus the number of local symmetries explains why there is only one scalar auxiliary field: $9(e_\mu^m) - 3(\text{Einstein}) - 3(\text{Lorentz}) = 3$ bosonic components, $6(\psi_\mu) - 2(\text{local susy}) = 4$ fermionic components. We shall use the following definition and identities

$$\varepsilon^{012} = -\varepsilon_{012} = 1, \quad e\gamma^{\mu\nu\rho} = -\varepsilon^{\mu\nu\rho}, \quad \varepsilon^{\mu\rho\sigma} \gamma_\mu = -\gamma^{\rho\sigma}, \quad \gamma^{\mu\nu} \gamma_\nu = 2\gamma^\mu. \quad (85)$$

We use the 1.5 order formalism, meaning that ω_μ^{mn} is determined by solving its own algebraic field equations [32]. As a consequence one never needs to vary (the e_μ^m or ψ_μ in) ω_μ^{mn} when one varies the bulk action, but only the e_μ^m and ψ_μ which are explicitly shown in (83). However, varying the e_μ^m and ψ_μ term in ω_μ^{mn} leads to boundary terms which we shall analyze. The local susy transformations read

$$\begin{aligned} \delta e_\mu^m &= \frac{\kappa}{2} \bar{\epsilon} \gamma^m \psi_\mu, & \delta \psi_\mu &= \frac{1}{\kappa} D_\mu \epsilon + \frac{1}{2\sqrt{2}} \gamma_\mu S \epsilon \\ \delta S &= \frac{1}{2\sqrt{2}} \bar{\epsilon} \gamma^{\mu\nu} (D_\mu \psi_\nu)^{\text{cov}}, \end{aligned} \quad (86)$$

where

$$(D_\mu \psi_\nu)^{\text{cov}} = D_\mu \psi_\nu - \frac{1}{2\sqrt{2}} \kappa S \gamma_\nu \psi_\mu \quad (87)$$

is the supercovariant field strength of the gravitino. (The local susy variation of (87) contains no term with $\partial_\mu \epsilon$). The spin connection is also supercovariant and given by [32]

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e) + \frac{\kappa^2}{4} (\bar{\psi}_\mu \gamma^m \psi^n - \bar{\psi}_\mu \gamma^n \psi^m + \bar{\psi}^m \gamma_\mu \psi^n) \quad (88)$$

where

$$\begin{aligned} \omega_{\mu mn}(e) &= \frac{1}{2} e_m^\nu (\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu}) \\ &\quad - \frac{1}{2} e_n^\nu (\partial_\mu e_{m\nu} - \partial_\nu e_{m\mu}) - \frac{1}{2} e_m^\rho e_n^\sigma (\partial_\rho e_{p\sigma} - \partial_\sigma e_{p\rho}) e_\mu^p. \end{aligned} \quad (89)$$

Again the local susy variation of (88) contains no terms with a derivative of ϵ .

We choose Gaussian coordinates, so that $x^{\mathbf{n}}$ is the arc length along geodesics normal to the boundary, and $x^{\mathbf{i}}, x^{\mathbf{j}}$ are coordinates in the surface. Then

$$g_{\mathbf{nn}} = 1, \quad g_{\mathbf{nj}} = g^{\mathbf{nk}} = 0. \quad (90)$$

The normal vector is given by $n^\mu = g^{t\mu}/(g^{tt})^{1/2}$ in a general coordinate system, but in Gaussian coordinates

$$n^\nu = n_\nu = \delta_\nu^{\mathbf{n}}. \quad (91)$$

We can use this equation to extend n to a vicinity of the boundary. We also choose the Lorentz indices in such way that the vielbein fields are block-diagonal

$$e_{\mathbf{n}}^N = 1, \quad e_{\mathbf{n}}^a = 0, \quad e_{\mathbf{j}}^N = 0, \quad e_{\mathbf{j}}^a \text{ arbitrary} \quad (92)$$

where N and a are flat indices corresponding to the curved indices \mathbf{n} and \mathbf{j} . The use of this special coordinate system considerably simplifies calculations (see Appendix A for technical details). We stress that we do not suppose that the variations of the fields also satisfy (90) and (92). The vielbein defined by (92) is not invariant under the diffeomorphism and Lorentz transformations on the boundary. Therefore, one has to be very careful when using the coordinate system defined above. For example, symmetry variations of normal components of the fields contain both the normal components of usual symmetry variations *and* also the terms with symmetry variations of the normal vector itself.

First we consider the model without the York-Gibbons-Hawking boundary term. From the Euler-Lagrange variational equations one finds the following BC

$$(\delta_e \omega_{\mathbf{j}}^{mn}) e_m^{\mathbf{j}} e_n^{\mathbf{n}}|_{\partial\mathcal{M}} = 0 \quad (93)$$

and

$$\left[-\frac{e}{\kappa^2} (\delta_\psi \omega_{\mathbf{j}}^{mn}) e_m^{\mathbf{j}} e_n^{\mathbf{n}} - \frac{1}{2} \varepsilon^{\mathbf{ijn}} \bar{\psi}_{\mathbf{i}} \delta \psi_{\mathbf{j}} \right] |_{\partial\mathcal{M}} = 0. \quad (94)$$

The last term in (94) vanishes whenever $P_+ \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ or $P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$. Then both the variation $\delta \omega_{\mathbf{j}}^{mn} e_m^{\mathbf{j}} e_n^{\mathbf{n}}$ induced by the variation of the vielbein, and also the variation $\delta \omega_{\mathbf{j}}^{mn} e_m^{\mathbf{j}} e_n^{\mathbf{n}}$ induced by the variation of the gravitino, must vanish at the boundary. Hence *any* variation of this component of the spin connection vanishes. One has to resolve the BC on the components of the spin connection to obtain corresponding BC for the vielbein and for the gravitino. We postpone this task for a while and continue to analyze the BC on the spin connection.

The boundary terms due to a local susy variation read

$$n_\nu \left[\frac{1}{2\kappa} \bar{\epsilon} D_{\mathbf{i}} \psi_{\mathbf{j}} \varepsilon^{\mathbf{ijn}} - \frac{1}{4\sqrt{2}} \bar{\psi}_{\mathbf{i}} \gamma_{\mathbf{j}} \epsilon S \varepsilon^{\nu\mathbf{ij}} - \frac{e}{\kappa^2} e_m^\rho e_n^\nu \delta_\epsilon \omega_\rho^{mn} \right] \quad (95)$$

where n is an outward pointing unit vector. The first variation is due to partial integration of $\delta \bar{\psi}_\mu = \partial_\mu \bar{\epsilon} + \dots$ which is needed to produce a curvature (this curvature subsequently cancels against another curvature which is obtained by varying the explicit vielbeins in the Einstein-Hilbert action). The second variation is due to partial integration of $\partial_\rho \delta \psi_\sigma \sim \partial_\rho (\gamma_\sigma S \epsilon)$, and the last term is due to varying all fields in the spin connection.

We are still considering the case without Gibbons-Hawking boundary term. In that case we know from the BC (30) of the linearized theory that $\bar{\psi}_{\mathbf{i}} \gamma_{\mathbf{j}} \epsilon$ is nonvanishing, hence the auxiliary field must satisfy the BC

$$S|_{\partial\mathcal{M}} = 0. \quad (96)$$

As we explained before the last term in (95) vanishes. The first term can be solved by inserting $P_+ + P_- = I$ into it and using that $P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0$ and $\bar{\epsilon} P_+|_{\partial\mathcal{M}} = 0$. In Gaussian coordinates P_\pm commute with the ordinary derivative $\partial_{\mathbf{i}}$, so that one only finds the BC

$$\omega_{\mathbf{i}}^{aN}|_{\partial\mathcal{M}} = 0. \quad (97)$$

If one now uses the BC from the field equations according to which both $\delta_e \omega_{\mathbf{j}}^{aN}|_{\partial\mathcal{M}} = 0$ and $\delta_\psi \omega_{\mathbf{j}}^{aN}|_{\partial\mathcal{M}} = 0$, one sees that the BC (97) is consistent: *any* variation of this BC also vanishes. Next we compare (97) with (124) to see that the following BC holds

$$K^{\mathbf{ij}}|_{\partial\mathcal{M}} = 0. \quad (98)$$

The gravitino part of the condition (97) yields for the contractions with $e_a^{\mathbf{i}}$

$$0 = (e_a^{\mathbf{i}} \omega_{\mathbf{i}}^{aN}(\psi))|_{\partial\mathcal{M}} = \left[\frac{\kappa^2}{2} \bar{\psi}_{\mathbf{j}} \gamma^{\mathbf{j}} \psi^N - \frac{\kappa^2}{4} \bar{\psi}_{\mathbf{j}} \gamma^n \psi^{\mathbf{j}} \right] |_{\partial\mathcal{M}}. \quad (99)$$

The last term in this equation vanishes, while the first term yields $\bar{\psi}_{\mathbf{j}} \gamma^{\mathbf{j}} P_+ \psi^N|_{\partial\mathcal{M}} = 0$, or

$$P_+ \psi_N|_{\partial\mathcal{M}} = 0. \quad (100)$$

Then all of (97) vanishes.

Finally we resolve the BC on the spin connection obtained above and close the orbit of the BC (98) and (100) to obtain the following set of BS

$$\partial_{\mathbf{n}} e_{\mathbf{j}}^a|_{\partial\mathcal{M}} = 0, \quad \delta e_{\mathbf{n}}^a|_{\partial\mathcal{M}} = 0, \quad \delta e_{\mathbf{j}}^N|_{\partial\mathcal{M}} = 0 \quad (101)$$

$$P_- \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad P_+ \psi_{\mathbf{n}}|_{\partial\mathcal{M}} = 0, \quad (102)$$

$$P_+ \partial_{\mathbf{n}} \psi_{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad P_- \partial_{\mathbf{n}} \psi_{\mathbf{n}}|_{\partial\mathcal{M}} = 0, \quad (103)$$

$$S|_{\partial\mathcal{M}} = 0. \quad (104)$$

The BC on the parameters are as follows

$$\xi^{\mathbf{n}}|_{\partial\mathcal{M}} = 0, \quad \partial_{\mathbf{n}} \xi^{\mathbf{j}}|_{\partial\mathcal{M}} = 0, \quad (105)$$

$$P_- \epsilon|_{\partial\mathcal{M}} = 0, \quad P_+ \partial_{\mathbf{n}} \epsilon = 0, \quad (106)$$

$$\lambda^{aN}|_{\partial\mathcal{M}} = 0, \quad \partial_{\mathbf{n}} \lambda^{ab}|_{\partial\mathcal{M}} = 0. \quad (107)$$

Here λ^{mn} are parameters of the Lorentz transformations. Symmetry considerations do not require any restrictions on $e_{\mathbf{n}}^N$. However, if one wishes to impose a BC on $e_{\mathbf{n}}^N$, this condition must be Neumann

$$\partial_{\mathbf{n}} e_{\mathbf{n}}^N|_{\partial\mathcal{M}} = 0, \quad (108)$$

since the Dirichlet condition violates susy.

By using the formulae from Appendix B one can easily check that the boundary conditions (101) - (104) are closed under the action of all local

symmetries provided the transformation parameters satisfy (105) - (107). The BC (97) and (98) are satisfied as well.

Let us now turn to the case when the Gibbons-Hawking boundary term is added to the action, and when one chooses therefore the Gibbons-Hawking BC for the gravity fluctuations

$$\delta g_{ij}|_{\partial\mathcal{M}} = 0. \quad (109)$$

This case was studied by Luckock and Moss [6]. We shall use many results from that paper. No locally invariant set of BC was presented in [6]. By modifying slightly the analysis of that paper we show that such a set does not exist.

Obviously, the BC on the spin connection (97) are not satisfied in the presence of the York-Gibbons-Hawking term. Also in the presence of this term the diffeomorphism invariance of the action implies the BC

$$\xi^n|_{\partial\mathcal{M}} = 0 \quad (110)$$

on the parameter ξ^μ . Consistency requires that also

$$0 = \delta_\xi g_{ij}|_{\partial\mathcal{M}} = (\xi_{i;j} + \xi_{j;i} - 2\Gamma_{ij}^n \xi_n)|_{\partial\mathcal{M}}, \quad (111)$$

where the colon denotes covariant differentiation with the Christoffel symbol constructed from the metric of the boundary. Together with the BC (110) the equation (111) tells us that ξ_j is a Killing vector on the boundary. There is at most a finite number of Killing vectors which generate rigid symmetries. In this section we are interested in local symmetries, so that we assume for simplicity that there are no Killing vectors on $\partial\mathcal{M}$. Consequently, we have the following BC

$$\xi_i|_{\partial\mathcal{M}} = 0. \quad (112)$$

Closure of the susy algebra then requires $\bar{\epsilon}_2 \gamma_\mu \epsilon_1|_{\partial\mathcal{M}} = 0$ for all indices μ . This condition yields

$$\epsilon|_{\partial\mathcal{M}} = 0. \quad (113)$$

If one wishes to couple this system to a spin 1 field, our linearized analysis shows (see eq. (15)) that the gravitino must satisfy

$$P_- \psi_n|_{\partial\mathcal{M}} = 0. \quad (114)$$

Even if no spin 1 fields are present, a more tedious analysis of hermiticity properties of the fluctuation operators [6] also requires the same condition (114) of the gravitino¹³. Consistency now requires

$$0 = \delta_\epsilon P_- \psi_{\mathbf{n}}|_{\partial\mathcal{M}} = P_- \partial_{\mathbf{n}} \epsilon|_{\partial\mathcal{M}}, \quad (115)$$

where we used (113). Therefore, one has *both* Dirichlet and Neumann BC on $P_- \epsilon$. This clearly excludes local susy transformations on the boundary.

We must stress that this conclusion is valid for local BC only. If one allows for nonlocal BC, one can easily resolve the contradiction we have found above. However, it still remains an open question whether one can find a closed consistent orbit of BC with nonlocal BC.

7 Conclusions and comments

In this article we have determined the complete consistent set (“the orbit”) of BC for supergravity models, which maintains local susy even at the boundaries. Violation of local susy by boundaries may not be fatal, it may perhaps even be welcome, but we have studied when local susy remains unbroken. We have worked completely at the classical level; at the quantum level, boundary term may also be needed to remove infrared divergences [33]. The renormalization group flow affects the boundary conditions [20].

Our main result is that local susy of the BC in supergravity requires vanishing extrinsic curvature $K^{\mathbf{ij}}|_{\partial\mathcal{M}} = 0$. The surfaces with zero extrinsic curvature are called totally geodesic. Such surfaces contain geodesics connecting any two points belonging to them. It is interesting to note that totally geodesic surfaces are also minimal, i.e. they are solutions of the classical equations of the bosonic p -branes.

We considered massless fields in the text. For massive fields there are differences. Consider the massive spin 0 - spin 1/2 system with $\delta_\epsilon \lambda = \delta_\epsilon (m = 0)\lambda + m(S + iP)\epsilon$. For the BC $P_- \epsilon = 0$, $P_+ \lambda|_{\partial\mathcal{M}} = \partial_{\mathbf{n}} S|_{\partial\mathcal{M}} = P|_{\partial\mathcal{M}} = 0$. one finds from $P_\pm \delta_\epsilon \lambda = 0$ the BC

$$(\partial_{\mathbf{n}} \pm m)S = 0. \quad (116)$$

since $P_\pm \gamma^{\mathbf{n}} = \pm P_\pm$ if $(\gamma^{\mathbf{n}})^2 = 1$. Similarly for interacting theories there are differences (see section 3).

¹³This condition follows from the equations (6.48), (6.49) and the definition (6.20) of [6]. Note that compared to that paper we have interchanged the role of P_+ and P_- .

In the case of pure gravity, gauge or BRST invariant BC for the graviton were constructed in [34, 35] (see [18] for an overview). It is interesting to note that the BC obtained in [34] either contain tangential derivatives of the fields on the boundary, or admit noncovariant gauges only. The sets BC of [35], which do not depend on tangential derivatives were obtained with some restrictions on the extrinsic curvature of the boundary. In the presence of tangential derivatives in BC, quantum loop calculations become extremely difficult.

Local BC for supergravity were considered in [10, 6, 12, 18]. All these papers started with the Gibbons-Hawking condition (2) on the gravity fluctuations. Therefore, it was not possible to obtain a fully locally supersymmetric set of BC. There exists a great variety of types of BC which are being used in quantum field theory (see reviews [36, 37]). Here we did not consider nonlocal BC of the Atiyah-Patodi-Singer type or other exotic conditions. Nonlocal BC in supergravity were studied in [17, 18, 12], but no locally susy orbit of such BC was found. In many applications it is desirable to have the Gibbons-Hawking BC for the gravity fluctuations, or at least the York-Gibbons-Hawking boundary term in the action (see, e.g., [7, 38]). Since our no-go result is valid for local BC only, one should probably reconsider the nonlocal option again.

Globally supersymmetric asymptotic conditions were constructed by Breitenlohner and Freedman [39] who considered gauged supergravity in AdS. They obtained two sets of the asymptotic conditions which correspond¹⁴ to our sets (24) and (30). Later Hawking [40] suggested an additional requirement to choose between these two sets. He required that the space-time approaches anti-de Sitter space sufficiently fast at infinity that the asymptotic group of motion of the space-time is the AdS group $O(3, 2)$. In this case there exist asymptotically supercovariant constant spinors which generate asymptotic global susy transformations. If one now demands that the space-time remains asymptotically AdS, one is led to the asymptotic conditions which correspond to the second set (30). This is precisely the set of BC which was selected in section 6 as preserving *local* susy on the boundary.

¹⁴One has to note that asymptotic conditions are not the same as BC. Nevertheless, one can map one into the other by identifying the fields which vanish fast at the infinity with the fields which satisfy Dirichlet BC. We also like to mention here related works [41] where boundary terms in the AdS space and their properties with respect to rigid symmetry transformations were analysed. Various choice of BC in AdS were discussed recently in [42, 27].

Therefore, certain problem with supersymmetries in the first set (24) were noted long ago [40] though not for BC but rather for asymptotic conditions on AdS. This is particularly remarkable given large interest to supergravities and AdS in general and to the asymptotic conditions on graviton in particular [7].

Matching conditions on a brane which restrict the extrinsic curvature were studied in supergravity only recently by Moss [43]. That paper, however, did not analyse the closure of the set of matching conditions under all symmetry transformations.

Note added in proof

If one removes some of our requirements, e.g. if one does not require the consistency of the boundary conditions with the equations of motion, then the York-Gibbons-Hawking action can be made compatible with local susy [44]. We are grateful to Dmitry Belyaev for explaining this point to us. We also like to mention the work [45] where boundary terms for the Lovelock gravity were studied. For a discussion of boundary terms which remove second order derivatives from all fields in the Hilbert-Einstein action, see [46].

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A Extrinsic curvature

The extrinsic curvature is defined by the relation $K_{\mu\nu} = (g_{\mu\rho} \mp n_\mu n_\rho)(g_{\nu\sigma} \mp n_\nu n_\sigma) D^\rho n^\sigma$. The induced metrics $g_{\mu\nu} \mp n_\mu n_\nu$ yield projection operators $g^{\mu\rho}(g_{\rho\nu} \mp n_\rho n_\nu) = \delta^\mu_{\nu} \mp n^\mu n_\nu$ if $g^{\mu\nu} n_\mu n_\nu = \pm 1$. We continue with the upper sign. Since $\partial_\rho(n^\mu n_\mu) = 0 = 2n_\mu D_\rho n^\mu$, it can be simplified to

$$K_{\mu\nu} = n_{\nu;\mu} - n_\mu n^\rho n_{\nu;\rho}. \quad (117)$$

In Gaussian coordinates $n_\nu = (1, 0, \dots, 0)$, $g_{\mathbf{nn}} = g^{\mathbf{nn}} = 1$, and $g_{\mathbf{ni}} = g^{\mathbf{ni}} = 0$, hence

$$K_{\mathbf{ij}} = -\Gamma_{\mathbf{ij}}^{\mathbf{n}} = \frac{1}{2}\partial_{\mathbf{n}}g_{\mathbf{ij}}; \quad K^{\mathbf{ij}} = -\frac{1}{2}\partial_{\mathbf{n}}g^{\mathbf{ij}}, \quad (118)$$

and all other components of $K_{\mu\nu}$ vanish.

Next we give several useful relations between variations of the metric and the normal vector. We suppose that before the variation the “background” metric (denoted by $\bar{g}_{\mu\nu}$) is Gaussian. The full metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$ is not Gaussian, of course, but the varied normal $n^\mu = \bar{n}^\mu + \delta n^\mu$ is still perpendicular to the surface, $n_{\mathbf{i}} = 0$, and normalized to unity, $n^\mu g_{\mu\nu} n^\nu = 1$. The variation of $g^{\mu\nu}$ can be expressed in terms of $\delta g_{\mu\nu}$ as follows

$$\delta g^{\mathbf{ij}} = -\bar{g}^{\mathbf{ik}}\bar{g}^{\mathbf{jl}}\delta g_{\mathbf{kl}}, \quad \delta g^{\mathbf{in}} = -\bar{g}^{\mathbf{ik}}\delta g_{\mathbf{kn}}, \quad \delta g^{\mathbf{nn}} = -\delta g_{\mathbf{nn}}. \quad (119)$$

Under arbitrary variations of the metric the normal varies as follows

$$\begin{aligned} \delta n^{\mathbf{n}} &= -\frac{1}{2}\delta g_{\mathbf{nn}}, & \delta n^{\mathbf{j}} &= -\bar{g}^{\mathbf{jk}}\delta g_{\mathbf{nk}}, \\ \delta n_{\mathbf{n}} &= \frac{1}{2}\delta g_{\mathbf{nn}}, & \delta n_{\mathbf{j}} &= 0. \end{aligned} \quad (120)$$

It is straightforward to prove that the variation of extrinsic curvature reads

$$\begin{aligned} \delta K_{\mathbf{nn}} &= 0, \\ \delta K_{\mathbf{nj}} &= \bar{K}_{\mathbf{j}}^{\mathbf{k}}\delta g_{\mathbf{nk}}, \\ \delta K_{\mathbf{ij}} &= -\frac{1}{2}\bar{K}_{\mathbf{ij}}\delta g_{\mathbf{nn}} - \frac{1}{2}[(\delta g_{\mathbf{in}})_{;\mathbf{j}} + (\delta g_{\mathbf{jn}})_{;\mathbf{i}} - \partial_{\mathbf{n}}\delta g_{\mathbf{ij}}]. \end{aligned} \quad (121)$$

The colon denotes covariant differentiation with the Christoffel symbol defined by the metric $\bar{g}_{\mathbf{jk}}$.

The equations (119) - (121) do not use any boundary conditions and, therefore, can be differentiated with respect to $x^{\mathbf{n}}$. Note, that we have extended the normal vector and, consequently, the extrinsic curvature to outside the boundary. In our coordinate system

$$\bar{\Gamma}_{\mathbf{nn}}^{\mathbf{n}} = \bar{\Gamma}_{\mathbf{nn}}^{\mathbf{k}} = \bar{\Gamma}_{\mathbf{nk}}^{\mathbf{n}} = 0, \quad \bar{\Gamma}_{\mathbf{jk}}^{\mathbf{n}} = -\bar{K}_{\mathbf{jk}}, \quad \bar{\Gamma}_{\mathbf{nk}}^{\mathbf{j}} = \bar{K}_{\mathbf{k}}^{\mathbf{j}} \quad (122)$$

B Torsion

Consider the bosonic part of the spin-connection. It can be defined through the vielbein equation

$$D_\mu e_\nu^m = \partial_\mu e_\nu^m - \Gamma_{\mu\nu}^\rho e_\rho^m + \omega_\mu^m{}_p(e)e_\nu^p = 0, \quad (123)$$

From this equation we have the following components of the connection $\bar{\omega}_\mu(\bar{e})$ in the adapted coordinate system (90) - (92):

$$\begin{aligned}\bar{\omega}_{\mathbf{n}}^{aN} &= 0, & \bar{\omega}_{\mathbf{n}}^{ab} &= -\bar{e}^{jb}\partial_{\mathbf{n}}\bar{e}_j^a - \bar{K}_{\mathbf{j}\mathbf{i}}\bar{e}^{ja}\bar{e}^{ib}, \\ \bar{\omega}_{\mathbf{i}}^{aN} &= \bar{K}_{\mathbf{ij}}\bar{e}^{ja}, & \bar{\omega}_{\mathbf{i}}^{ab} &= -\bar{e}^{jb}\partial_{\mathbf{i}}\bar{e}_j^a + \bar{\Gamma}_{\mathbf{ij}}^{\mathbf{k}}\bar{e}_k^a\bar{e}^{jb}.\end{aligned}\quad (124)$$

This formulae imply that on the boundary all components of $\bar{\omega}_\mu(\bar{e})$ except for $\bar{\omega}_{\mathbf{j}}^{ab}$ vanish for the BC (98).

Next we study the gravitino part of the connection $\omega_\mu^{mp}(\psi)$. One can prove that our boundary conditions (102) yield

$$\omega_{\mathbf{n}}^{ab}(\psi)|_{\partial\mathcal{M}} = 0, \quad \omega_{\mathbf{j}}^{aN}(\psi)|_{\partial\mathcal{M}} = 0. \quad (125)$$

Consider first

$$\omega_{\mathbf{n}}^{ab}(\psi) = \frac{\kappa^2}{4} (\bar{\psi}_{\mathbf{n}}\gamma^a\psi^b - \bar{\psi}_{\mathbf{n}}\gamma^b\psi^a + \bar{\psi}^a\gamma_{\mathbf{n}}\psi^b). \quad (126)$$

On the boundary, the following identities hold

$$\begin{aligned}\bar{\psi}_{\mathbf{n}}\gamma^a\psi^b &= \bar{\psi}_{\mathbf{n}}\gamma^a(P_+ + P_-)\psi^b = \bar{\psi}_{\mathbf{n}}\gamma^a P_+\psi^b = \bar{\psi}_{\mathbf{n}}P_-\gamma^a\psi^b = 0, \\ \bar{\psi}^a\gamma_{\mathbf{n}}\psi^b &= \bar{\psi}^a\gamma_{\mathbf{n}}(P_+ + P_-)\psi^b = \bar{\psi}^a\gamma_{\mathbf{n}}P_+\psi^b = \bar{\psi}^a P_+\gamma_{\mathbf{n}}\psi^b = 0.\end{aligned}\quad (127)$$

The first equality in (126) is now obvious, the second one can be demonstrated in a similar manner.

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